From bar diagrams to letter-symbolic algebra: a technology-enabled bridging

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Abstract
In the Singapore primary school Mathematics curriculum, students are taught the model method that uses bar diagrams to visualize the problem structure in a given word problem. When these students progress to secondary school, they learn the algebraic way of solving word problems. Studies (e.g. Ng et al.) have shown that poor bridging of students from the use of bar diagrams to the use of letter-symbolic algebraic methods can hinder their learning of algebra. We design a software tool ‘ALGEBAR’ to scaffold the learning of the algebraic process, especially the formulation of equations to support a pedagogy that seeks to help students transit from bar diagrams to algebraic methods. In this paper, we report a cycle of evaluation of the intervention pedagogy by examining a class that uses the software tool as part of a holistic intervention package. This evaluation is in the context of an overall design research approach that sought to address complex problems in real classroom contexts in collaboration with practitioners and to integrate design principles with technological affordances to render plausible solutions. Our findings show that the software tool can be an important enabler of the bridging process.

Keywords
conceptual change, intelligent feedback, learning of algebra, Mathematics, pedagogy and technology integration, scaffolding.

Introduction
Singapore students in primary schools (from ages 9–12) are taught to solve letter-symbolic algebraic word problems using rectangular bar diagrams. In Singapore, the practice of using these types of diagrams is known among teachers and students as the ‘bar-modelling’ method or the ‘model method’, which aims to depict the situation as presented in the given word problem. The bars represent known and unknown quantities from the information given in the problem. Students are expected to solve for the unknown by analysing the relationships among the bars. This approach was introduced more than two decades ago with the specific aim of helping primary school students solve structurally complex word problems (Kho 1987). The efficacy of the ‘model method’ may be as a result of its reliance on iconic (visual) representation as an intermediary in a three-step approach based on Bruner’s (1966) Concrete-Pictorial-Abstract (Enactive-Iconic-Symbolic) framework that is adopted by Singapore’s Mathematics curriculum. For example, students without knowledge of letter-symbolic algebra could use the bar diagrams to solve the following word problem (Ng et al. 2007):

A cow weighs 150 kg more than a dog. A goat weighs 130 kg less than the cow. Altogether the three animals weigh 410 kg. What is the weight of the cow?

In Fig 1, the model shows the relationships between the weights of three animals. Here the bar representing the dog is used as the ‘generator’ (Bednarz & Janvier 1996) of the model drawing. This bar is an unknown unit. The representations for the other two animals are
based on that for the dog. The differences in weight between the animals and the total weight of the three animals are the known values. The model drawing becomes a pictorial equation representing the statement that ‘3 bar units plus 170 correspond to 410’. The value of 1 unit can be found by deducing that 3 units correspond to 240 (i.e. 410 - 170), and hence 1 unit is 80 kg. The weight of the cow is the sum of 1 unit and 150. The pictorial equation representation is equivalent to an algebraic equation, \[ x + (x + 150) + (x + 20) = 410, \]
where \( x \) represents the weight of the dog.

When these students progress to secondary school, they learn the algebraic way of solving word problems. Studies (e.g. Ng et al. 2006) have shown that poor bridging of students from the use of bar diagrams to the use of letter-symbolic algebraic methods can hinder their learning of algebra. Figures 2 and 3 show two Secondary 1 students’ solutions to a word problem. The students have started to learn letter-symbolic algebra and thus use ‘\( x \)’ to label the unknown, but their problem-solving process could not qualify as a truly algebraic method, which would have entailed the following steps: (1) formulation of an appropriate equation; (2) algebraic manipulation of the equation; (3) algebraic solution of the equation; and (4) translation from the algebraic solution to answer the question. Indeed, the students employed a literal translation from the models to algebraic equations, and some of their equations would have been meaningless without the models. What we would like to do is to eventually empower students of algebra with the ability to solve problems algebraically without reference to models.

As one can see, it is indeed a challenge in designing appropriate curricula and pedagogy to help students transit to letter-symbolic algebraic methods. A new pedagogy has been developed (Kho 2007). Working in tandem with the Singapore Ministry of Education, we designed and developed a software tool, ‘ALGEBAR’, to scaffold the learning of the algebraic process, especially the formulation of equations.

The outline of this paper is as follows: we review the existing research work based on understanding the mathematical word problem-solving process, and discuss the elements of the bridging pedagogy, the design of the software tool, as well as the results of an empirical study on the use of the tool in a school.

What do we know of from existing research?

From existing research, we know that the main issues in the learning of letter-symbolic algebra include the following:

- Students encounter difficulties in forming equations from the word sentences (Wollman 1983).
- It is important for students to develop a structural understanding of letter-symbolic algebraic equations (Kieran 1989).
- Students have the tendency to calculate known numbers that is against the flow of letter-symbolic algebraic problem-solving (Stacey & MacGregor 2000).

Stacey and MacGregor (2000) sought to investigate the reasons for students’ difficulties in solving word problems using letter-symbolic algebra apart from those already well-known in the literature (e.g. grammar and context). It is not easy to move from understanding a word problem to formulating an equation. There are cognitive discontinuities when changing from arithmetic to letter-symbolic algebraic thinking. First, students have to operate with unknowns instead of numbers. Second, there is a difference between procedural and structural views of expressions and equations. Third, the process of solving problems is different. In contrast, it is students’ second nature to solve problems by forward calculation. In their study, they found out that students either do not try the letter-symbolic algebraic path or get deflected from it. They are more comfortable working with known instead of unknown quantities. Among the non-letter-symbolic algebraic ways used are arithmetic reasoning, trial and error (random guessing, sequential guessing, guess-check-improve), and \textit{post hoc} writing of formulas (an arithmetic method disguised as letter-symbolic algebraic in which the unknown is straightaway made the subject of the equation).
Q1: There are 50 children in a dance group. If there are 10 more boys than girls, how many girls are there?

\[ \begin{align*}
\text{Girls} & \quad \begin{array}{c}
\times \\
\rightarrow 10
\end{array} \\
\text{Boys} & \quad \begin{array}{c}
\times + 10
\end{array} \\
\end{align*} \]

Let the number of girls be \( x \).

\[ \begin{align*}
\text{Girls} & \rightarrow x \\
\text{Boys} & \rightarrow x + 10 \\
50 - 10 & = 40 \\
40 \div 2 & = 20 \\
\text{There are 20 girls}.
\end{align*} \]

Q2: An amount of money $120 is shared among 3 persons A, B and C. If A receives $20 less than B, and B receives 3 times as much as C, how much money does C receive?

\[ \begin{align*}
\text{A} & \quad \begin{array}{c}
\rightarrow 3x
\end{array} \\
\text{B} & \quad \begin{array}{c}
\rightarrow 3x - 20
\end{array} \\
\text{C} & \quad \begin{array}{c}
\rightarrow 20
\end{array} \\
\end{align*} \]

Let \( C \) receive \( x \).

\[ \begin{align*}
\text{B} & \rightarrow 3x - 20 \\
\text{A} & \rightarrow 3x - 20 + 20 = 140 \\
7x & \rightarrow 120 + 20 = 140 \\
x & \rightarrow 140 \div 7 = 20 \\
\text{C receives} & \rightarrow 20 \text{.}
\end{align*} \]

Although a model with a similar structure can be used to solve arithmetic as well as letter-symbolic algebra word problems, the solution to the letter-symbolic algebra word problem is more challenging. In the seminal study by Ng (2003), 79% of 145 Secondary 2 (about 14 years old) students in Singapore reported that they found the bar diagrams useful as it aided them in visualizing the problem. Their teachers also reported that students who have been taught letter-symbolic algebra often continue to rely on the bar diagrams to solve similar types of word problems. In Ng’s study, some students’ solutions showed a mixture of the two methods. Ng notes that while this can be seen as a sign of mental flexibility, persistence in using the bar diagrams can be a hindrance to the acquisition of letter-symbolic algebra. This latter view may explain why secondary school teachers are tempted to dismiss the ‘model method’ by asking Secondary 1 and 2 students to abandon the method at the outset when they are taught formal letter-symbolic algebra.

Ng et al. (2007) report findings from four studies that investigated various issues related to the use of the bar
diagrams and its relationship to letter-symbolic algebra. In one of the studies, teachers found that both the model method and the symbolic method were related as they were able to use the drawn models to construct letter-symbolic algebraic equations and *vice versa*, although the former sequence was less effortful than the latter. Using neuroimaging technique and working with adults who were equally competent with the two methods, another study showed that both the model method and symbolic method activated very similar neuroanatomical structures, suggesting that both these methods recruited similar cognitive processes. It follows that we can argue that the model method is algebraic in nature, except that it uses rectangular bars instead of letter symbols. The findings from these two studies may inform pedagogical solutions to the bar diagrams versus letter-symbolic algebra dilemma (Ng et al. 2007). If we follow the ‘conceptual change’ framework of Vosniadou and Vamvakoussi (2005), then teachers would be ill-advised to ask students to inhibit or completely jettison the model method. Rather, they should help students see connections among the strategies that they learn (Ng et al. 2007). Students need to appreciate the importance of learning the symbolic method because it is the language of letter-symbolic algebra that is predominant and valued in higher Mathematics, and because post-Vietan algebra is able to express general solutions for the unknown in terms of given parameters (Harper 1987), a feature that the model method does not afford.

Compared with other primary Mathematics curricula (Cai et al. 2005), Singapore’s policy to teach primary school students a more visual approach to problem-solving has something to offer the world in terms of enabling students to solve problems generally not thought to be tractable for their age. Indeed, Singapore students have consistently performed well in the international Trends in Mathematics and Science Study (TIMSS). In TIMSS 2003, Singapore’s fourth and eighth graders (equivalent to Primary 4 and Secondary 2, respectively) were ranked first in Science and Mathematics among 49 countries (TIMSS 2008). In the USA, more than 100 school districts and schools have adopted or have experimented with the Singapore Math approach for their elementary schools (Viadero 2000; Hoff 2002). However students’ persistent continued use of the forward calculations encouraged by the model method at the secondary school level demands a re-conceptualization of how the letter-symbolic algebra as a problem-solving tool should be taught.

**Design of bridging pedagogy**

In designing the scaffolding process to support the bridging from bar diagrams to letter-symbolic algebra, from a cognitive perspective, secondary school students use mixed schemas or ‘synthetic models’ (Vosniadou & Vamvakoussi 2005) when learning to solve word problems. After at least 6 years of primary school, the arithmetic and non-letter algebraic methods of solution appear to have become ‘reified’ in students. Then in secondary school, the students are taught letter-symbolic algebra. Thus, when faced with a word problem, a student would activate schemas to attempt solving the problem. They are more prone to activate the more familiar ‘primary-school’ schemas or allow these schemas to interfere with the new letter-symbolic algebraic schemas.

In the course of observing Secondary 1 students solve word problems during the course of our pilot studies, we have seen many instances where they have an inherent tendency to want to do forward calculations. Whatever prior learning a student has strongly influences the subsequent learning. Most students are able to understand that the unknown is a specific number being referred to. However, it is not enough simply to tell the student that the unknown corresponds to a bar. It is also not enough to just let them form known expressions in a sequence, form the unknown units into one single expression and then equate the two together. Problems in which the unknown appears once in an equation can often be solved arithmetically by calculation in reverse (e.g. \[
\frac{36}{x - 3} = 6
\]). Students should be comfortable formulating equations where the unknown can appear on both sides of an algebraic equation (e.g. \(10 + 5(x - 4) = 3x\)). To do that, students need to know all the meanings of the various bar diagram symbols and map them to an equation. For example, in bar diagrams, the vertical brace means a sum. So in the design of instruction, students should be taught how to translate a diagram connecting several rows into an equation, with several corresponding terms summed to another expression. In other words, students need to be taught the explicit structural homomorphism between bar model representation and algebraic equations.
The process of solving equations requires structural understanding of letter-symbolic algebra. Students need to be comfortable manipulating these structures and transforming the equation formed in a series of deductions until the unknown appears as the subject of the equation. Teachers need to understand this tendency in students to calculate as impeding the learning of letter-symbolic algebraic method and adopt strategies to overcome this. But teachers can help by first using models as a means of scaffolding meaning-making. However, this scaffold should fade away once students acquire the habit of formulating and solving equations. Teachers should then introduce gradually harder problems, especially those that can only be solved by using letter-symbolic algebra. Only then students will appreciate the power of letter-symbolic algebra and still be able to link it to their previous learning.

**Design of technology to support the pedagogy**

A key design consideration in the intervention approach is to integrate technological support from the very beginning as part of the pedagogy design. We postulate that we can harness the affordances of technology (e.g. individualized, immediate and meaningful feedback; graphical layout suggestive of the algebraic problem-solving process; prescriptive interaction instead of prescriptive interaction allowing for multiple solution paths) to guide the student. A pivotal step is to ask the student to formulate an algebraic equation that involves the unknown ‘x’ as part of the expression. This will be the main objective of the software tool called ALGEBAR. A subsequent step is to solve the algebraic equation for ‘x’: as ‘x’ may or may not be the answer asked for in the problem, the last step is to map the solution to the answer being asked. The tool provides the process and the structure for the student to draw the model, receive feedback on the model drawn, formulate the algebraic equation that represents the relationships as depicted in the model, solve the equation and receive feedback on the solution.

The following principles undergird the design of ALGEBAR, which is meant to help scaffold the construction of an algebraic equation from a given word problem:

1. The student has available a graphic toolkit to construct a model; he or she has the option (but not the obligation) to draw the model first in order to scaffold the construction of the algebraic equation. The student can choose to formulate the equation straight-away if so desired.

2. The interaction is *proscriptive* rather than *prescriptive*. Instead of telling students what steps to do first and what to do next, students can do anything they like subject to consistency constraints. The system should be able to check for various forms of consistencies, including: model self-consistency, equation self-consistency, model-to-equation consistency, algebraic working self-consistency, equation-to-algebraic working consistency and algebraic solution consistency with the formed equation. The tool allows a virtually infinite range of models and equations, except when consistency constraints are violated, whereupon the system feedbacks to the student on what has gone wrong.

3. The tool should be able to provide positive and negative feedback as the student is working through the problem. A Linear Algebra Inference Engine (LAIE) was implemented for this purpose. Although not as powerful as a Computer Algebra System, the LAIE is able to solve linear equations and determine whether a system of linear equations is solvable but indeterminate, uniquely solvable or unsolvable.

Figure 4 shows how a problem is presented along with the scaffolding tools available to the students. Students can then use these tools to construct models for the following types of algebraic problems: percentages, ratios, fractions and whole numbers.

As a way of presenting the design of the software, we shall recount the process of students working on different algebraic problems in class during the pilot intervention in schools. We captured sessions of these students working on the problems using a recording tool called MORAE (TechSmith Corp., Okemos, MI, USA) which records the computer screen and camera video. Given the problems, students had the option of building the model or defining the variable needed to solve the problem.

From the data collected, the students chose to build the model first. Figure 5 shows the initial model drawn by a student. The software checked the model drawn for inconsistencies and detected one such inconsistency in the relationship between the ‘Malay’ and the ‘Chinese’ bars. It then prompted the student to check the model...
drawn and make corrections to the model. The student then revised the model as shown in Fig 6.

When the student clicked on ‘Check Model’, the software did not detect any inconsistencies or errors with the model, and the student proceeded to formulate the equation. The student then defined the variable by letting $x$ be the number of Indian students. He then proceeded to formulate the equation $3x + 254 = 875$. *AlgeBar* checked the equation and detected that the equation was valid and consistent with the model drawn. It then proceeded to prompt the student to solve the algebraic equation to obtain the solution for $x$ in the working space as shown in Fig 7.

Besides checking for consistencies within the model, and between the model and the equation formed, *AlgeBar* also checks the algebraic solution for mathematical consistencies by using the LAIE. It provides feedback on each step of the solution of the letter-symbolic algebraic equation (see Figs 8–10). *AlgeBar* checks the syntax of the well-formulated equation in one variable, as well as whether each line of the algebraic solution is mathematically consistent with the previous lines (see Fig 8).

Because students are prone to interject arithmetic forward calculations in their solution, the software will look for instances of this (typically an expression...
without a variable, without an ‘=’ sign or with double ‘=’ signs in two different contexts) and flag this out to the students as they are working on the algebraic manipulations. Figure 9 shows how the LAIE prompted the student that there is an inconsistency in his algebraic solution as he has interjected arithmetic calculations in his algebraic solution, i.e. using ‘=’ in two different contexts on the same line.

In another question, a student constructed the model and formed the equation as shown in Fig 10. The software used the LAIE to solve the student-constructed equation and detected that the answer was mathematically inconsistent with the correct answer for the problem. AlgebAr then prompted the student to check the model drawn and the equation formed for errors.

AlgebAr does not check for consistency between the model drawn and the problem. It only checks for consistency between the algebraic solution provided and the final answer to the problem from the question database. When students drew incorrect models and proceeded to solve the problems and specify their solu-
tion, ALGEBAR detected that the solutions provided were incorrect and prompted the students to check their answers. Although ALGEBAR does not implicitly tell users where or what was wrong with their solution, nevertheless, the line-by-line checking has helped the students in backtracking their solutions and tracing where their mistake was. Eventually, the students realized that their models were incorrect. They corrected their models, modified the equation formed and solved the problems.

**Research design**

Our overarching research questions are to study the differential effects of the bridging pedagogy together with the use of ALGEBAR as a scaffolding tool on students’ ability to formulate and solve mathematical problems letter-symbolic algebraically, and to do design research to iteratively improve the pedagogy, the instructional materials and the design of the tool. Our research questions are: does our bridging pedagogy help students to...
better learn letter-symbolic algebra? How does the enacted bridging pedagogy help students to better learn letter-symbolic algebra? How does technology (ALGEBAR) enhance the bridging pedagogy? Figure 11 shows our iterative design research cycle that incorporates learning theory, instructional package and software development. We adopted a design research approach in our school-based work to address complex problems in real classroom situations in collaboration with practitioners and to integrate design principles with technological affordances to render plausible solutions. Our goal is to conduct rigorous and reflective inquiry to test and refine innovative learning environments, as well as to refine new learning design principles (Brown 1992; Collins 1992). In early cycles of the intervention as enacted in classroom implementation, we have focused very much on teacher professional development, integration of the tool into a holistic lesson plan, design of the software tool and design of the problems. Because of lack of space, we will not report these aspects further in this paper. In the next section, we report the results of a formative empirical evaluation of the intervention package to help us have a better feel of the efficacy of our approach.

Research methodology

We worked with a neighbourhood school in evaluating the efficacy of the ALGEBAR software with its associated approach of introducing algebraic methods to students trained in the model methods to solving word problems. The school identified two classes in Secondary 1 for purposes of this research. The intervention took place over a period of 4 weeks in Secondary 1 Mathematics. Ideally, a research study would benefit from a longer intervention period. However, the curriculum hours dedicated to the teaching of algebra to solve word problems in Secondary 1 is only 4 weeks, and this places a cap on the intervention period.

The unit taught was the use of algebra to solve problems in whole numbers, fractions, ratios and percentages. Prior to this, students were already taught algebraic manipulations such as expansions and factorization, and the concepts for ratio and percentage. In this unit, the objectives were to teach the students to define an algebraic variable and to formulate algebraic equations before proceeding to solve the problems.

For data collection, pre- and post-intervention clinical interviews were conducted on three students from the experimental class. Pre- and post-intervention interviews were conducted with the teachers teaching the control and experimental classes. At the end of 4 weeks, a post-intervention test was conducted for both control and experimental classes. The post-test was scored strictly on algebraic methods and working as the
Learning outcome was for students to approach word problems with algebraic methods instead of model methods.

**Experimental class and control class profiles**

The control class and experimental class were closely matched in many aspects. This was deliberate so as to eliminate unnecessary variables. Each class comprises 34 students, with relatively similar spreads of natural mathematics ability as measured by their Primary School Leaving Examination’s (PSLE) Mathematics grades. In Singapore’s education system, every student at the end of 6 years of primary (elementary) education sits for a national examination called PSLE. Their performance at PSLE serves as a kind of ranking, which is a strong determinant of which secondary (middle) school they will go to. The control class has a PSLE average of 214.6 while the intervention class has 214.4, out of a maximum possible score of 300. The control class has a PSLE Mathematics grade average of 3.5 and standard deviation (sd) of 0.67 while the intervention class has a grade average of 3.6 and sd of 0.78, based on a simple computation using $A^* = 5$, $A = 4$, $B = 3$, $C = 2$ and $D = 1$. Both classes were also taught by teachers of similar experience and ability.

**Teaching approaches in experimental class versus control class**

The teachers teaching the control class and experimental class shared a number of similarities in their teaching approaches. They were the only two teachers assigned to teach Secondary 1 express classes in the school. They used a common set of lesson plans, except when the experimental class teacher conducted laboratory
lessons using AlgeBAR. They used materials from the same textbook and concurred on their teaching strategies before going for classroom lessons. Both teachers relied heavily on the textbook in class. They would refer the students to specific pages, highlight concepts, worked examples and get the students to attempt the ‘Try It’ problems found in the text before reviewing the solutions. Thereafter, exercises from the text were assigned as homework, and students commenced their work immediately. Here is where the similarities between the control class and experimental class teachers end.

In terms of teaching pedagogy, both teachers used the model method to bridge into algebra. This was also the strategy employed in the textbook and illustrated in worked examples. However, over the course of the intervention, it was observed that, on the one hand, the experimental class teacher consciously sought to wean the students off the model method by defining variables and formulating equations without referring to models. On the other hand, the control class teacher continued to draw models and referred to them as he defined variables and formulated equations. At this juncture, it is important to clarify that any improvement found in the experimental class would not be attributed solely to the use of AlgeBAR. We see the software as an enabler and an essential part of a holistic technology-enabled pedagogy package.

The experimental class has an hour in the computer laboratory using AlgeBAR for the 4 weeks of intervention. A typical lesson would have the teacher demonstrate using one or two examples before the students were tasked to work in pairs on randomly generated questions. The teacher would walk around to monitor students’ progress and correct their mistakes. It was observed that for ordinary problems involving whole numbers, the students were competent in using AlgeBAR to draw models and were able to define the variables and formulate the main equations. When the class moved on to problems involving fractions and ratios, students were observed to have difficulty drawing models for more difficult questions and were inevitably unable to define the variables and main equations. When students were finally able to construct the models, they somehow seemed to be in a better position to define the variables.

To sum up, the control class and experimental class used the same lesson plans for classroom lessons, similar textbook material, and approached the teaching of algebra using model drawing. However, the experimental class has 4 h of classroom lessons substituted with 4 h of technology-enabled lessons in the laboratory.

**Manifestations of algebraic skills in experimental class versus control class**

After 4 weeks of intervention, a post-test was conducted for both the experimental and control classes. The post-test scored the students strictly on algebraic methods and workings. In other words, the post-test score is an indication of the manifestations of algebraic skills acquired by the students. An independent-samples t-test conducted to compare the post-intervention test scores for control and experimental classes yield the following results (Table 1). The test scores for the experimental class ($M = 50.35$, $sd = 19.77$) were significantly higher than for the control class [$M = 30.82$, $sd = 16.71$; $t(66) = -4.399, P < 0.001$]. The magnitude of the differences in the means [means’ difference $= -19.529$, 95% confidence interval (CI) $-28.394$ to $-10.665$] was very large (eta-squared $= 0.23$). According to Cohen (1988, pp. 284–287), eta-squared $\geq 0.14$ indicates a large effect (Table 1).

To control for the confounding effect of the students’ natural ability in Mathematics, we conducted a one-way between-groups analysis of covariance to compare the post-test scores of the two classes while using the students’ PSLE Mathematics grade as covariate. After adjusting for the students’ PSLE Mathematics grades, the experimental class (adjusted $M = 49.93$) still performed significantly better than the control class (adjusted $M = 31.25$) in the post-test [$F(1, 65) = 19.9$, $P < 0.001$, partial eta-squared $= 0.23$]. Twenty-three per cent of the variance in the post-test scores is explained by the intervention. This confirms that the experimental class has performed significantly better than the control class after intervention (Table 2).

We also analyse the extent of algebraic methods and workings demonstrated by the students in the post-test. The post-test comprises 10 compulsory questions, each of which carries 5 marks. The scoring criterion is based strictly on algebraic methods and workings. Two-thirds of the marks were allocated to defining of variables and formulation of equations. As an example, consider post-test question 4: There are 680 boys and girls in a school. Three-fourths of the number of boys is equal to 2/3 of
the number of girls. How many boys are there? The marking scheme gave one mark for defining the number of boys to be $x$, one mark for defining the number of girls to be $680 - x$, or alternatively $\frac{9}{8}x$, and one mark for formulating the main equation $\frac{3}{4}x = \frac{2}{3}(680 - x)$, or alternatively $x + \frac{9}{8}x = 680$. The remaining two marks were awarded for algebraic manipulation and an accurate answer.

A $\chi^2$ test (with Yates’ Continuity Correction) indicates that the use of algebraic method is significantly associated with the experimental class compared with its use in the control class [$\chi^2(1, 680) = 67.6, P < 0.001$, $\phi = 0.32$] (Table 3). The experimental class attempted the post-test using 81.8% algebraic methods and workings, compared with 51.8% by the control class.

**Observations and inference of post-test data**

As the students had 4 years of training in using the bar diagrams (model) method to solve word problems in their primary schools, they are very comfortable with using this approach. In the 4 weeks, they had lessons that introduced them to algebraic methods. Their exposure to algebraic methods is limited in that period, and hence we note that they are still coping with approaching the word problems in the post-intervention test using an algebraic approach. This is understandable as students have only been exposed to algebra for a month compared with 4 years of training in model drawing in primary schools. The independent-samples $t$-test shows that the performance of the experimental class in using the algebraic methods is significantly better than that of the control class. Furthermore, even when controlling for the students’ natural mathematics ability, the performance of the experimental class in using algebraic methods is still significantly better than that of the control class.

**Observations from pre-intervention and post-intervention clinical interviews with students**

The pre-intervention clinical interview with students comprises a quiz of two questions and a short interview. Three students of different abilities (high, average, weak) were chosen from the experimental classes for the interview. The same three students underwent a

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<th>Standard deviation</th>
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<td>34</td>
<td>30.82</td>
<td>16.712</td>
<td>-4.399***</td>
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<td>34</td>
<td>50.35</td>
<td>19.771</td>
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***$P < 0.001$.

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<td>2.957</td>
<td>19.904***</td>
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<tr>
<td>Experimental class</td>
<td>34</td>
<td>49.93$^1$</td>
<td>2.957</td>
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$^1$Covariates are evaluated at PSLE Math = 3.57.

***$P < 0.001$.

PSLE, Primary School Leaving Examination.

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<td>Algebraic (%)</td>
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<th>Method</th>
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<td>Control class</td>
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<td>Experimental class</td>
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***$P < 0.001$. 

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similar structured post-intervention interview. The three students were hand-picked by the class teacher to represent the three distinct ability groups present in the class. We were counting on the professional judgement of the teacher and his profound knowledge of the class to identify students who epitomize each ability level.

The first observation from the pre-intervention quiz indicates that the students are familiar with the model method, especially in solving routine questions. We illustrate here a student’s solution and trace his progression to post-intervention (Fig 12). In the post-intervention quiz question 1, the same student demonstrated his ability to define the variables algebraically and was able to partially formulate the algebraic equations.

The second observation indicates that when the problem demands more sophisticated model drawing, as in pre-intervention quiz question 2, students are unable to translate the problem into model, and thus, are unable to proceed further. Only one student was able to draw the model for this question, and it was only partially labelled too. Equipping the students with algebra will help them solve problems that do not lend themselves to model drawing easily. We traced the transition of the above student who now encountered difficulties drawing the challenging model in pre-intervention quiz question 2 to his ability to use algebra in post-intervention quiz question 2 (Fig 13).

The third observation from post-intervention interview, and supported by the post-test, indicates that after 4 weeks of intervention, the stronger student is able to define variables and formulate equations, the average student is able to define variables and formulate equations for routine questions, but the weaker student is only able to define variables correctly (Table 4).

All the three students interviewed associate algebra with ‘using letters to substitute for unknowns’. They have modest confidence in its use as it is a relatively new topic. They believe that algebra is a powerful tool that can help to solve problems efficiently, but they have not fully grasped its concepts. When given a problem, only the stronger student (one out of three) indicates

**Pre-intervention Quiz Question-1**
Barbara has 50 more stickers than Alan. Charlie has 30 stickers less than Barbara. Altogether, the three of them have 190 stickers. How many stickers does Barbara have?

Solution by BL

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

190

1 unit

190 – 50 – 20 = 120
3 units = 120
1 unit = 40
Barbara = 40 + 50 = 90

**Observation:** The student was able to solve the problem using model method.

**Post-intervention Quiz Question-1**
Megan has 3 times as many songs in her computer as Sarah. On Sunday, their father came across 60 new songs in the internet. He emailed 25 to Megan and the remaining to Sarah. As a result, Megan now has twice as many songs as Sarah. How many songs did Megan have in the beginning?

Solution by BL (after intervention)

Let the amount Sarah have be x.

Before:
Sarah = x
Megan = 3x

After:
Sarah = x + 25
Megan = 3x + 25

3x + 25 = x + 25 (mistake!)

**Observation:** The student made mistakes here.
Table 4. Analysis of competency levels attained by students of different abilities.

<table>
<thead>
<tr>
<th>Questions</th>
<th>Student 1 (considered weak by the teacher)</th>
<th>Student 2 (considered average by the teacher)</th>
<th>Student 3 (considered strong by the teacher)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Post-test question 5 (easy)</td>
<td>Let number of ‘Free trip to Sentosa’ be $x$. Use model next.</td>
<td>Let number of ‘Free trip to Sentosa’ be $x$. Number of ‘$5 voucher’ = 5$x 5$x + x = 30% 6$x = 300 Proceed to solve for $x$.</td>
<td>Let number of ‘Free trip to Sentosa’ be $x$. No. of ‘$5 voucher’ = 5$x 5$x + x = 30% 6$x = 300 Proceed to solve for $x$.</td>
</tr>
<tr>
<td>Post-test question 4 (difficult)</td>
<td>Let the number of boys be $x$. $\frac{3}{4}x = \frac{2}{3}x$ (wrong!) Let the number of girls be $x$. $\frac{3}{4}y = \frac{2}{3}x$ (wrong!) $\frac{3}{4}y + \frac{3}{3}x = 680$ (wrong!) Use model next.</td>
<td>Let the number of boys be $x$. $\frac{3}{4}x = \frac{2}{3}x$ (almost there!) Let the number of girls be $x$, boys be $y$. $\frac{3}{4}y = \frac{2}{3}x$ (right!) Proceed to solve for $x$.</td>
<td>Let the number of girls be $x$. Boys $\rightarrow$ 680 $- x$ $\frac{3}{4}(680-x) = \frac{2}{3}x$ Proceed to solve for $x$. Let the number of boys be $x$. Girls $\rightarrow$ $\frac{3}{2} \times \frac{3}{4}x = 9\frac{9}{8}x 9\frac{8}{8}x + x = 680$ Proceed to solve for $x$.</td>
</tr>
<tr>
<td>Post-test question 9 (difficult)</td>
<td>Let number of chicken burgers be $x$. Unable to proceed or attempt using ratio method.</td>
<td>Let number of chicken burgers be $x$. Attempt to solve using model or ratio method.</td>
<td>Let number of fish burger be $x$. Chicken burgers $= 3x$. After lunch: Fish burgers $= x$ Chicken burgers $= \frac{1}{2}x \frac{3x - 40}{x} = \frac{1}{2}$ Proceed to solve for $x$.</td>
</tr>
<tr>
<td>Observations</td>
<td>This student is not able to define the variables correctly.</td>
<td>This student is able to define the variables correctly and to correctly formulate equations for routine problems.</td>
<td>This student is able to define the variables correctly and to formulate equations for both routine and challenging problems.</td>
</tr>
</tbody>
</table>
Pre-intervention Quiz Question 2
Jeremy and Ali collect fighter plane cards. When Jeremy gives 15 of his cards to Ali, he has 4 times as many cards as Ali. When Jeremy gives another 5 cards to Ali, he has three times as many cards as Ali. How many cards does Ali have?

Solution by BL

<table>
<thead>
<tr>
<th>J</th>
<th>15</th>
<th>20</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>15</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Later

<table>
<thead>
<tr>
<th>J</th>
<th>20</th>
<th>20</th>
<th>15</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>15</td>
<td>5</td>
<td></td>
<td></td>
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</tbody>
</table>

Observation: The student had difficulty drawing the model. Although his approach was correct, lack of clarity in interpretation and ability to construct led to mistakes in the model.

This hindered further progression of his solution.

Post-intervention Quiz Question 2
Jack and Annie have a collection of stickers, and they share them in the ratio 5 : 7. Their father gave each of them 150 more stickers, and the ratio became 3 : 4. Find the number of stickers that Annie has?

Solution by BL (after intervention)

Let the number of stickers Annie has be $x$.

Before:

<table>
<thead>
<tr>
<th>Jack</th>
<th>$\frac{5}{7}x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annie</td>
<td>$x$</td>
</tr>
</tbody>
</table>

After:

$3(x + 150) = 4\left(\frac{5}{7}x + 150\right)$

$x + 450 = \frac{20}{7}x + 600$

$x = 1050$

Observation: If the student has attempted to draw and solve using the model, it will be quite challenging. Fortunately, he has learned algebra and solved the problem elegantly.

readiness to use algebra, and the other students are likely to fall back on model method or other heuristics.

Interviews with teachers

Both teachers shared the opinion that algebra is one of the major obstacles for students entering secondary school. In primary schools, students were taught visual approaches such as model method and guess-and-check. The transition from visual approach to abstract approach such as algebra is difficult. Students are unable to make sense of symbolic expressions. They are unable to integrate arithmetic into algebraic working as required in problem-solving. According to control class teacher, Mr C., ‘Although they are actually able to do arithmetic and algebraic manipulations efficiently on separate slates, difficulty arises when they are integrated into problem-solving. Students have the greatest difficulty formulating the main equations’.

The experimental class teacher, Mr E., observed that ‘AlgeBAR has benefited students in some ways, regardless of whether it was intended for that purpose or not’. It helps students to understand the problem better and faster. This is because it facilitates model drawing, the process of which helps students to visualize the problem faster and better. As a result, students in the experimental class seem to be able to formulate the main equations better than students in the control class. This was evident in the post-test scores. On the surface, this observation of AlgeBAR facilitating model drawing might seem...
contradictory, as the assumption was to wean students off their prior acquired skills at model drawing. Students are generally proficient in model drawing involving regular problems, but as the complication of a problem increases, the model drawing is likely to stall and put a hold to the solution. This was supported by observations made during the laboratory lessons when the experimental class moved on to more challenging problems involving fractions and ratios. Students were observed to encounter difficulty drawing models and were unable to define the variables and main equations. When they were finally able to construct the models, many of them were also able to define the variables.

ALGEBAR also helps students to check for consistency in the model and equations. This checking mechanism provides scaffolds for the students, letting them know if they are on the right track. It enables the students to explore and attempt the questions with less help from the teacher.

However, Mr E. felt that ‘AlgeBAR does not lend itself very well to problems involving fractions, ratios and percentages. For example, \( \frac{x}{x+40} = \frac{5}{6} \) is difficult to express in models using AlgeBAR.’ As a result, using ALGEBAR to solve such problems becomes non-intuitive as compared with using algebra directly. Such difficulty may also indicate to students that the model method is not the only way or the optimum way to solve all problems.

The teachers felt that laboratory lessons have much to offer compared with regular classroom lessons. Because of the settings, the use of computers or the interactivity that arises, students seem more enthusiastic and motivated. Random generation of questions in ALGEBAR exposes the class to a richer and greater variety of questions. Students seem to have benefited from a combination of classroom lessons and laboratory lessons using ALGEBAR.

Summary and conclusion

In this paper, we explicate a Mathematics learning challenge unique to Singapore schools. Our primary school students are taught using bar diagrams (among other heuristics) to solve word problems, and they have become very competent at it. The bright students typically have no problem in switching to letter-symbolic algebraic methods in secondary school. Other students seem to hold on to the bar diagrams as a crutch, which hampers them from developing efficacy in letter-symbolic algebraic methods. From our preliminary studies, we have evidence that these students are confounding their letter-symbolic algebraic solutions with arithmetic forward calculations. Our observations are consistent with Ng’s (2003) study and also with Stacey and MacGregor’s (2000) observation that students tend to like to do forward calculations with known numbers instead of manipulating algebraic equations as structures. Our observations and Ng’s (2003) observations differ from Stacey and MacGregor’s (2000) observations in that using bar diagrams is the predominant non-letter-symbolic algebraic method in the Singapore context.

We discuss the pedagogy intervention and, in particular, the software design in ALGEBAR. We describe an empirical study of an intervention we did with a secondary school and discuss results that show the efficacy of the approach. The research is part of a bigger ongoing research design programme that is not merely a software development cycle. We hope to develop an accompanying instructional package girded by an instructional theory for the teaching and learning of algebra. Future research areas include these aspects of the enactment of the pedagogy in the classroom: working with different cohorts of students and with different teachers, professional development of teachers, the interweaving of classroom-based discussions and software-based activities, and collaborative work between pairs of students as they work together on a word problem in ALGEBAR.

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References


